

# Technique for computing the PDFs and CDFs of non-negative infinitely divisible random variables.

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## Abstract

We present a method for computing the PDF and CDF of a non-negative infinitely divisible random variable  $X$ . Our method uses the Lévy-Khintchine representation of the Laplace transform  $\mathbb{E}e^{-\lambda X} = e^{-\phi(\lambda)}$ , where  $\phi$  is the Laplace exponent. We apply the Post-Widder method for Laplace transform inversion combined with a sequence convergence accelerator to obtain accurate results. We demonstrate this technique on several examples including the stable distribution, mixtures thereof, and integrals with respect to non-negative Lévy processes. Software written to implement this method is available from the authors and we illustrate its use at the end of the paper.

## 1 Introduction

Let  $X$  be a non-negative random variable. The distribution of  $X$  is said to be infinitely divisible (ID) if for any positive integer  $n$ , we can find i.i.d. random variables  $X_{i,n}$ ,  $i = 1, 2, \dots, n$  such that

$$X \stackrel{d}{=} X_{1,n} + X_{2,n} + \dots + X_{n,n}.$$

For background, see [22], [3]. There are many examples of such distributions, including the gamma distribution, compound Poisson distributions, inverse Gaussian distribution and right-skewed stable distributions. These distributions are also central in the study of non-decreasing Lévy processes. In this paper, we give a method for numerically computing the probability density function (PDF) and cumulative distribution function (CDF) of a non-negative infinitely divisible random variable.

Our starting point is the Lévy-Khintchine (LK) formula ([4], [5],[21]), which in the case of non-negative ID random variables states that the Laplace transform of  $X$ ,

$$\psi(\lambda) \equiv \mathbb{E}e^{-\lambda X} = e^{-\phi(\lambda)}, \quad \lambda > 0, \quad (1)$$

has an exponent  $\phi(\lambda)$  which is called the Laplace exponent, which can be written as

$$\phi(\lambda) = a\lambda + \int_0^\infty (1 - e^{-\lambda u})\Pi(du). \quad (2)$$

Here,  $a \geq 0$  is a shift and  $\Pi$  is a measure on  $(0, \infty)$  which satisfies

$$\int_0^\infty (1 \wedge x)\Pi(dx) < \infty. \quad (3)$$

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Since we are interested primarily in PDFs and CDFs, we will assume that  $a = 0$  throughout, since a positive  $a$  only shifts the PDF/CDF. Let  $f_X(x)$  and  $F_X(x)$  denote the PDF and CDF of  $X$ , respectively (we will promptly drop the subscript  $X$  on these functions when it is clear from the context). The Laplace transforms of the PDF and CDF can be easily obtained from the LK formula (we will use a tilde to denote a Laplace transform):

$$\psi(\lambda) = \tilde{f}(\lambda) = \int_0^\infty e^{-\lambda x} f(x) dx,$$

and a simple application of Fubini's theorem implies

$$\Psi(\lambda) \equiv \tilde{F}(\lambda) = \int_0^\infty e^{-\lambda x} \left( \int_0^x f(y) dy \right) dx = \frac{1}{\lambda} \int_0^\infty e^{-\lambda y} f(y) dy = \frac{\psi(\lambda)}{\lambda}. \quad (4)$$

Thus, obtaining  $f$  and  $F$  is a matter of inverting a Laplace transform. Generally, this task is not easy. Typically it is done by complex integration of the Laplace transform (see [2]), which can be difficult if the integrands are slowly-decaying, oscillatory functions. This causes many numerical integration methods to converge slowly. Here, we apply a different method of Laplace inversion known as the *Post-Widder (PW) method* ([1], Theorem 2 or [7], section VII.6). It is based on the fact that under weak conditions on a function  $f$ , we have

$$f(x) = \lim_{k \rightarrow \infty} \frac{(-1)^{k-1}}{(k-1)!} \left( \frac{k}{x} \right)^k \tilde{f}^{(k-1)} \left( \frac{k}{x} \right), \quad x > 0, \quad (5)$$

where  $\tilde{f}^{(k-1)}(k/x)$  denotes the  $(k-1)^{th}$  derivative of  $\tilde{f}$  evaluated at  $k/x$ . Thus, instead integrating the Laplace transform, we are taking arbitrarily high derivatives.

The obvious challenge in using this method is computing high derivatives of  $\tilde{f}$  in (5) to approximate the limit. Some methods have been developed to do this in general (see for instance, [10] or [2]), however most involve complex integration. The method we describe here for the case of infinitely divisible distributions is both easy to implement, and only involves (at worst) integration of a real valued non-negative exponentially decaying function. The method we use here combines both computation of a finite number of terms of the sequence in (5) and an numerical extrapolation method to approximate the limit, [8]. A similar method was used to compute mean first-passage times of Lévy subordinators in [23].

This paper is organized as follows. In Section 2 we give an overview of the approximation method used based on the (PW) formula. In Section 3, we outline the algorithm for finding  $f$  and  $F$ , as well as numerical issues that may arise in the computation. We test the method in cases where the PDF and CDF are known in closed form in Section 4. In Section 5 we apply our method to a collection of examples. The software for implementing the methods described here is freely available from the authors and its use is described in Section 6.

## 2 Post-Widder method with extrapolation

Given a continuous function  $g(x)$ ,  $x > 0$ , which is bounded as  $x \rightarrow \infty$ , we will denote its  $k^{th}$  PW approximation as

$$g_k(x) = \frac{(-1)^{k-1}}{(k-1)!} \left( \frac{k}{x} \right)^k \tilde{g}^{(k-1)} \left( \frac{k}{x} \right), \quad x > 0. \quad (6)$$

The fact that  $g_k(x) \rightarrow g(x)$  as  $k \rightarrow \infty$  can be seen by approximating  $g(x)$  by  $\mathbb{E}g(\bar{Y}_k)$ , where  $\bar{Y}_k = k^{-1}(Y_1 + Y_2 + \dots + Y_k)$ , and the  $Y_i$ 's are i.i.d. gamma random variables with mean  $x$  and variance  $x^2$  and then applying the law of large numbers (see Section VII.6 in [7]).

The convergence of  $g_k(x)$  to  $g(x)$  is slow in general. To illustrate this point, let  $g$  be a PDF  $f$  of an inverse Gaussian distribution for which the Laplace transform is given by  $\tilde{f}(\lambda) = \exp(-\lambda^{-1/2})$  (in this case

the density  $f(x)$  is known in closed form, see Section 4). In Figure 1, we plot the exact formula for  $f$  as well as  $f_k$  for  $k = 1, 10$  &  $50$ . Notice that even with 49 derivatives of  $\tilde{f}$ , the approximation is still poor. In fact, this is a general feature of the PW formula, as it has been shown ([10]) that the errors  $\epsilon_k(f; x) \equiv f(x) - f_k(x)$  have a power series expansion:

$$\epsilon_k(f; x) = \sum_{m=1}^{\infty} \frac{a_m(x)}{k^m} \quad (7)$$

where the coefficients  $a_m(m)$  are given by

$$a_m(x) = \sum_{j=1}^m f^{(j+m)}(x) x^{j+m} \frac{d(j+m, m)}{(j+m)!},$$

where  $d(j+m, m)$  are the associated Stirling numbers of the first kind ([18], Chapter 4, section 4). Thus, the convergence of  $f_k$  is in general  $O(1/k)$  corresponding to the  $m = 1$  term. This method alone is thus inadequate for computing the inverse Laplace transform to a high level of precision.

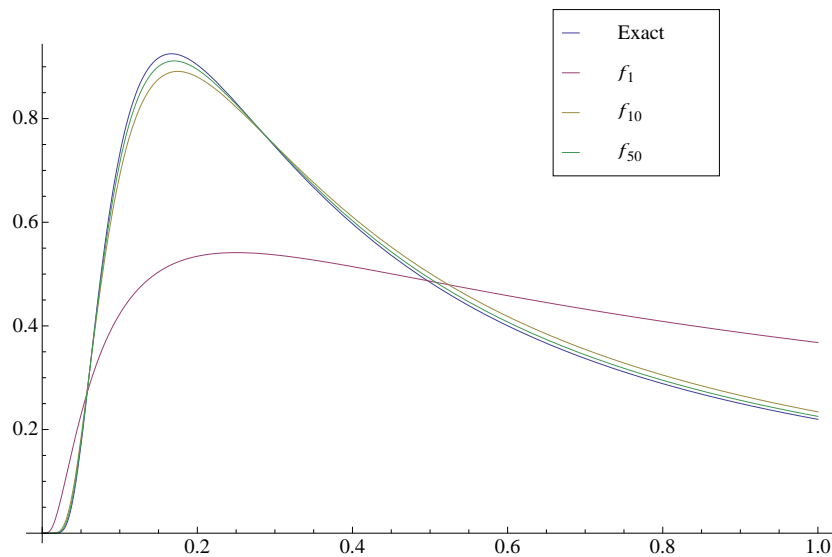


Figure 1: The exact form of  $f$  together with the PW approximations  $f_1$ ,  $f_{10}$  and  $f_{50}$

To obtain high precision, it is necessary to couple the PW approximations with a convergence acceleration method which extrapolates the limit in (5) based on a finite collection of terms in the sequence. Thus, our method for computing the PDF  $f$  at  $x > 0$  involves two components:

1. For some sequence  $k_1 < k_2 < \dots < k_N$ , compute the approximates  $f_{k_i}(x)$ .
2. Use the points  $(k_1, f_{k_1}(x)), \dots, (k_N, f_{k_N}(x))$  to extrapolate  $\lim_{k \rightarrow \infty} f_k(x)$ .

And for the CDF, replace  $f$  with  $F$  above. We focus first on step 1, which in sight of (6) involves computing high derivatives of  $\psi$  or  $\Psi$ . Next, we'll discuss two methods of convergence acceleration to address step 2.

**Remark:** Our method produces the best results if the PDF and/or CDF are smooth functions for  $x > 0$ . That being said, this method still produces useful results in the non-smooth case, however it may fail to

converge near points where the function lacks smoothness. This method should *not* be used for distributions which contain atoms, such as the Poisson distribution, or even distributions whose density lacks smoothness, such as a compound Poisson distributions with bounded jump distribution. For such distributions another method based on numerically solving a Kolmogorov-Feller forward equation, is more suitable, see [24].

## 2.1 Computing derivatives

### Derivatives of $\psi$

We begin by presenting two methods of computing derivatives of  $\psi(\lambda) = \exp(-\phi(\lambda))$ . The first method is based on the simple observation that

$$\psi'(\lambda) = -\phi'(\lambda)\psi(\lambda).$$

Thus, Leibnitz's formula implies that for any  $k \geq 1$ , the  $(k+1)^{th}$  derivative of  $\psi$  is given by

$$\psi^{(k+1)}(\lambda) = -\sum_{j=0}^k \binom{k}{j} \psi^{(j)}(\lambda) \phi^{(k+1-j)}(\lambda). \quad (8)$$

Notice that for  $\lambda$  fixed, the values of  $\psi(\lambda), \psi'(\lambda), \dots, \psi^{(k)}$  can be computed recursively using (8) if one has first computed  $\phi^{(j)}(\lambda)$  for  $j = 1, 2, \dots, k+1$ .

Alternatively, the  $k^{th}$  derivative of  $\psi(\lambda) = \exp(-\phi(\lambda))$  can be given directly in terms of the derivatives of  $\phi$  using Faà di Bruno's formula ([18], Chapter 2, section 8), which yields in our case

$$\psi^{(k)}(\lambda) = \frac{d^k}{d\lambda^k} e^{-\phi(\lambda)} = e^{-\phi(\lambda)} \sum \frac{k!}{m_1! m_2! \dots m_k!} \left( \frac{-\phi^{(1)}(\lambda)}{1!} \right)^{m_1} \dots \left( \frac{-\phi^{(k)}(\lambda)}{k!} \right)^{m_k},$$

where the sum runs over all  $m_1, \dots, m_k$  such that  $m_1 + 2m_2 + \dots + km_k = k$ . This can be expressed more simply in terms of the  $k^{th}$  complete Bell polynomial,  $B_k(x_1, x_2, \dots, x_k)$ , as

$$\psi^{(k)}(\lambda) = e^{-\phi(\lambda)} B_k(-\phi^{(1)}(\lambda), -\phi^{(2)}(\lambda), \dots, -\phi^{(k)}(\lambda)). \quad (9)$$

See [17] or [19]. Furthermore,  $B_k$  can be given in terms of the following  $k \times k$  determinant:

$$B_k(x_1, \dots, x_k) = \begin{vmatrix} x_1 & \binom{k-1}{1}x_2 & \binom{k-1}{2}x_3 & \binom{k-1}{3}x_4 & \dots & x_k \\ -1 & x_1 & \binom{k-2}{1}x_2 & \binom{k-2}{2}x_3 & \dots & x_{k-1} \\ 0 & -1 & x_1 & \binom{k-3}{1}x_2 & \dots & x_{k-2} \\ 0 & 0 & -1 & x_1 & \dots & x_{k-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -1 & x_1 \end{vmatrix}. \quad (10)$$

This provides another way for computing  $\psi^{(k)}$ .

Both methods above require the derivatives of the Laplace exponent  $\phi$ . Fortunately, these can be computed directly, since the LK formula implies that for any  $\lambda > 0$ ,

$$\phi^{(n)}(\lambda) = \begin{cases} \int_0^\infty (1 - e^{-\lambda u}) \Pi(du), & n = 0 \\ (-1)^{n+1} \int_0^\infty u^n e^{-\lambda u} \Pi(du), & n \geq 1 \end{cases} \quad (11)$$

Notice that passing derivatives through the integrals above is justified by the integrability assumption (3), since for any  $n \geq 1$ ,  $x^n e^{-\lambda x} \leq C_{n,\lambda}(1 \wedge x)$  uniformly on  $\{x > 0\}$  for some constant  $C_{n,\lambda} > 0$ . For many examples, these integrals have a closed form expression and can be computed easily. If a closed form is not available, many numerical integration methods are effective as these integrands are non-negative, exponentially decaying functions. In this case, it is important to use a small *relative* error tolerance, as  $\phi^{(n)}(\lambda)$  can become extremely small for large  $\lambda$ .

### Derivatives of $\Psi$

For  $\Psi$  defined in (4), we again apply Leibnitz's formula to (4) and obtain

$$\Psi^{(k)}(\lambda) = \tilde{F}^{(k)}(\lambda) = \frac{d^k}{d\lambda^k} \frac{\psi(\lambda)}{\lambda} = \sum_{j=0}^k \binom{k}{j} (-1)^j \frac{j!}{\lambda^{j+1}} \psi^{(k-j)}(\lambda),$$

When combined with the Post-Widder formula (6), we see a good deal of cancellation:

$$\begin{aligned} F_k(x) &= \frac{(-1)^k}{(k-1)!} \left(\frac{k}{x}\right)^k \tilde{F}^{(k-1)}(k/x) \\ &= \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{k}{x}\right)^k \Psi^{(k-1)}(k/x) \\ &= \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{k}{x}\right)^k \left[ \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \frac{j!}{(k/x)^{j+1}} \psi^{(k-1-j)}(k/x) \right] \\ &= \sum_{j=0}^{k-1} \frac{(-1)^{k+j-1}}{(k-j-1)!} \left(\frac{k}{x}\right)^{k-j-1} \psi^{(k-j-1)}(k/x) \end{aligned} \quad (12)$$

Note that this requires the values  $\psi(\lambda), \psi'(\lambda), \dots, \psi^{(n)}(\lambda)$ , which can be computed using (8) and (11) hence the recursive formula (8) is more convenient here than (9) since it computes all derivatives of  $\psi$ .

### Other useful derivatives

In some applications, the derivatives of the PDF are also of interest. Assuming the first  $q-1$  derivatives of  $f$  vanish at 0, i.e.  $f(0) = f'(0) = \dots = f^{(q-1)}(0) = 0$ , then  $f^{(q)}(x)$  has Laplace transform

$$\begin{aligned} \widetilde{f^{(q)}}(\lambda) &= \lambda^q \psi(\lambda) - \lambda^{q-1} f(0) - \lambda^{q-2} f'(0) - \dots - f^{(q-1)}(0) \\ &= \lambda^q \psi(\lambda), \end{aligned}$$

where  $\psi(\lambda) = \tilde{f}(\lambda)$ . In which case, Leibnitz's formula gives

$$\frac{d^k}{d\lambda^k} \widetilde{f^{(q)}}(\lambda) = \frac{d^k}{d\lambda^k} \lambda^q \psi(\lambda) = \sum_{j=0}^{q \wedge (k-1)} \binom{k}{j} \frac{q!}{(q-j)!} \lambda^{q-j} \psi^{(k-j)}(\lambda) \quad (13)$$

Thus, (8) and (11) can be applied to compute the derivatives of  $f$ .

## 2.2 Extrapolation

We want to compute the PDF  $f(x)$  or CDF  $F(x)$  by approximating the limit as  $k \rightarrow \infty$  in the Post Widder formula (5). By letting  $h = k^{-1}$ , (7) implies that  $f(x)$  can be written as

$$f(x) = f_k(x) + \sum_{m=1}^{\infty} a_m h^m \quad (14)$$

We shall truncate the error series  $\sum_{m=1}^{\infty}$  at  $m = N - 1$  and consider non-consecutive  $k_1 < k_2 < \dots < k_N$ . If  $h_i = k_i^{-1}$ ,  $i = 1, \dots, N$ , we can use the points  $(h_1, f_{k_1}(x)), (h_2, f_{k_2}(x)), \dots, (h_N, f_{k_N}(x))$  to estimate the first  $N - 1$  unknown coefficients  $a_m$ . Indeed, the (approximate)  $N \times N$  system

$$f_{k_i}(x) \approx f(x) - a_1 h_j - a_2 h_j^2 - \dots - a_n h_j^{N-1}, \quad j = 1, 2, \dots, N \quad (15)$$

can be solved for the  $N$  unknowns  $f(x), a_1, \dots, a_{N-1}$ . Since we are only interested in  $f(x)$ , it is enough to only compute the first row of the inverse of the matrix corresponding to the system (15). This is essentially done in the polynomial extrapolation method described below in order to obtain an accurate approximation of the limit  $\lim_{k \rightarrow \infty} f_k(x)$ . There is a vast literature on such extrapolation methods and the errors associated to them, see [11] for a review.

Below, we briefly review two techniques for approximating  $f(x)$  also used in [8]. We call these methods of extrapolation. The first is based on polynomial interpolation, and the second based on rational function (Padé) interpolation. Not surprisingly, the rational extrapolation provides faster convergence in many cases, however we found it more susceptible to numerical instability for larger  $N$  and  $k_N$ . Therefore, the “better” choice depends on the particular example and the desired accuracy.

### Polynomial extrapolation

Given the points  $(h_1, f_{k_1}(x)), (h_2, f_{k_2}(x)), \dots, (h_N, f_{k_N}(x))$ , the  $N - 1$  degree Lagrange polynomial  $P_N(x)$  which passes through these points is given by

$$P_N(h) = \sum_{i=1}^N \left( \prod_{j \neq i} \frac{h - h_j}{h_i - h_j} \right) f_{k_i}(x).$$

Thus, taking  $h = 0$  (or,  $k = \infty$ ) in this polynomial provides the approximation

$$f(x) \approx P_N(0) = \sum_{i=1}^N c_i f_{k_i}(x), \quad \text{where} \quad c_i = \prod_{j \neq i} \frac{-h_j}{h_i - h_j} = \frac{(-1)^{N-1} k_i^{N-1}}{\prod_{j \neq i} (k_j - k_i)}.$$

We thus use a linear combination of the approximations  $f_{k_i}(x)$  to obtain a more accurate approximation of  $f(x)$ .

### Rational extrapolation

As an alternative to polynomial extrapolation, one may instead fit a rational function to the points  $(h_1, f_{k_1}(x)), \dots, (h_N, f_{k_N}(x))$  of the form

$$R_N(h) = \frac{P_\mu(h)}{Q_\nu(h)},$$

where  $P_\mu(h) = \sum_{i=1}^{\mu} p_i h^i$  is a polynomial of degree  $\mu = \lfloor N/2 \rfloor$  and  $Q_\nu(h) = \sum_{i=0}^{\nu} q_i h^i$  is a polynomial of degree  $\nu = N - \lfloor N/2 \rfloor$ . Here, the coefficients  $p_i, q_i$  are chosen so  $R_N(h_i) = f_{k_i}(x)$ . When implementing this

method, these coefficients are not computed directly like in the polynomial extrapolation case, but instead  $R_N(0)$  is computed iteratively in a triangular array using the following recursive formulas (see [11], section 13)

$$R_{-1}^i = 0, \quad R_0^i = f_{k_i}(h_i), \quad i \geq 1$$

$$R_m^i = R_{m-1}^{i+1} + \frac{R_{m-1}^{i+1} - R_{m-1}^i}{h_i \Theta_m^i - h_{i+m}}, \quad \text{where} \quad \Theta_m^i = \frac{R_{m-1}^i - R_{m-1}^{i+1}}{R_{m-1}^{i+1} - R_{m-2}^{i+1}}.$$

We then take  $f(x) \approx R_N^1 = R_N(0)$  as our approximation since  $h = 1/k \rightarrow 0$  corresponds to  $k \rightarrow \infty$ . Notice that unlike the polynomial extrapolation, this transformation is *nonlinear* in the  $f_{k_i}(x)$ .

### Error bounds

One major drawback to using extrapolation techniques is that the error made in the approximation is difficult to bound analytically (see, for instance, Theorem 3 in [6] and equation (13) in [8]). One can however, obtain an asymptotic error bound for a rational or polynomial approximation. Let  $h_1 < h_2 < h_3 \dots$  be a sequence for which the approximant  $f_{k_i}(x)$  can be computed with  $k_i = 1/h_i$ . Also, let  $P_N(0)$  be the approximation of  $f(x)$  obtained by using polynomial extrapolation with  $h_1, h_2, \dots, h_N$ , with  $N \geq 2$  (all the following formulas also hold with rational extrapolation by replacing  $P_N(0)$  with  $R_N(0)$ ). In [6], Bulirsch and Stoer (BS) construct a *second* estimate of  $f(x)$ ,  $\tilde{P}_N(0)$ , with the property that

$$\lim_{N \rightarrow \infty} \frac{P_N(0) - f(x)}{\tilde{P}_N(0) - f(x)} = -1. \quad (16)$$

Thus, asymptotically,  $\tilde{P}_N(0)$  is as good an approximation to  $f(x)$  as  $P_N(0)$ , except that it approaches  $f(x)$  from the opposite direction. This second approximation is obtained using a linear combination of the form

$$\tilde{P}_N(0) = (1 + \alpha)P_{N+1}(0) - \alpha P_N(0), \quad (17)$$

and choosing the constant  $\alpha$  to change the sign of the leading term in the error  $P_N(0) - f(x)$ . BS show this constant  $\alpha$  is given by

$$\alpha = 1 + \frac{2}{\frac{h_1}{h_{N+1}} - 1}. \quad (18)$$

This allows us to construct a numerical bound on the relative error made in this method. From (16), we have

$$\frac{P_N(0) - \tilde{P}_N(0)}{f(x) - \tilde{P}_N(0)} = \frac{P_N(0) - f(x)}{f(x) - \tilde{P}_N(0)} + 1 \longrightarrow 2. \quad (19)$$

Fix any  $\eta \in (\frac{1}{2}, 1)$ . From (19) and since  $\eta > \frac{1}{2}$ , there exists  $N_0(\eta)$  large enough such that for  $N > N_0(\eta)$ ,  $|f(x) - \tilde{P}_N(0)| < \eta |P_N(0) - \tilde{P}_N(0)|$ . Using a similar argument, there also exists  $M_0(\eta)$  large enough such that for  $N > M_0(\eta)$ ,  $|P_N(0) - f(x)| < \eta |P_N(0) - \tilde{P}_N(0)|$ . Moreover, since  $P_N(0) \rightarrow f(x)$  and  $\eta < 1$ , we can pick  $N_1(\eta)$  large enough such that  $|P_N(0)/f(x)| < 1/\eta$  for all  $N > N_1(\eta)$ .

We thus can refine our estimate as  $f(x) \approx (P_N(0) + \tilde{P}_N(0))/2$ . If  $N > \max(N_0(\eta), M_0(\eta))$ ,

$$\left| \frac{P_N(0) + \tilde{P}_N(0)}{2} - f(x) \right| \leq \frac{|P_N(0) - f(x)|}{2} + \frac{|f(x) - \tilde{P}_N(0)|}{2} \leq \eta |P_N(0) - \tilde{P}_N(0)| \leq |P_N(0) - \tilde{P}_N(0)|,$$

and if  $N > \max(N_0(\eta), M_0(\eta), N_1(\eta))$ ,

$$\begin{aligned} \left| \frac{\frac{P_N(0) + \tilde{P}_N(0)}{2} - f(x)}{f(x)} \right| &\leq \frac{|P_N(0) - f(x)|}{2f(x)} + \frac{|f(x) - \tilde{P}_N(0)|}{2f(x)} \\ &\leq \eta \left( \frac{P_N(0)}{f(x)} \right) \frac{|P_N(0) - \tilde{P}_N(0)|}{P_N(0)} \\ &\leq \frac{|P_N(0) - \tilde{P}_N(0)|}{P_N(0)}. \end{aligned}$$

Therefore,

**Theorem 2.1** Suppose  $f(x) \neq 0$  and let  $P_N(0)$  be an approximation of  $f(x)$  obtained using either polynomial extrapolation or rational extrapolation, and let  $\tilde{P}_N(0)$  be defined as in (17). For  $N$  large enough, we have

$$\left| \frac{P_N(0) + \tilde{P}_N(0)}{2} - f(x) \right| \leq |P_N(0) - \tilde{P}_N(0)| \quad \text{and} \quad \left| \frac{\frac{P_N(0) + \tilde{P}_N(0)}{2} - f(x)}{f(x)} \right| \leq \frac{|P_N(0) - \tilde{P}_N(0)|}{P_N(0)}, \quad (20)$$

where the right hand sides converge to 0 as  $N \rightarrow \infty$ .

This provides a numerical bound on the absolute and relative errors of the approximation and suggests increasing  $N$  until either  $|P_N(0) - \tilde{P}_N(0)|$  or  $|P_N(0) - \tilde{P}_N(0)|/P_N(0)$  is smaller than a prescribed value.

### 3 Implementation

We begin with a description of the algorithm for computing the PDF and CDF based using the recursive formula (8) and the polynomial extrapolation method. This method is implemented in MATLAB<sup>1</sup> and Mathematica<sup>2</sup>. The method using rational extrapolation is also an option in the MATLAB implementation, but is not described here.

**Algorithm for computing the PDF  $f$  at  $x > 0$**

1. Choose the sequence  $1 = k_0 < k_1 < k_2 < \dots < k_N$  and a relative error tolerance  $\epsilon > 0$ . We found  $k_j = 10j$ , with  $N$  lying around 8 to be effective for  $\epsilon = 10^{-6}$ . Initialize the following arrays:

Variable	Size	Purpose
$D_\phi$	$(k_N - 1) \times N$	Holds derivatives of $\phi$
$D_\psi$	$k_N \times N$	Holds derivatives of $\psi$
$P$	$1 \times N$	Holds Post-Widder approximations of $f$
$f$	$1 \times N$	Holds extrapolated approximations of $f$

Set  $j = 1$ .

2. Compute  $(D_\psi)_{1,j}$  as

$$(D_\psi)_{1,j} = \psi(k_j/x)$$

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<sup>1</sup>MATLAB is computational software package developed by The Mathworks

<sup>2</sup>Mathematica is a computational software package developed by Wolfram Research



3. For  $i = k_{j-1}, \dots, k_j - 1$ , compute the  $i^{th}$  row of  $D_\phi$  as

$$(D_\phi)_{i,j} = \begin{cases} \phi^{(i)}(k_j/x) & i \leq k_j \\ 0 & \text{otherwise} \end{cases}, \quad j = 1, 2, \dots, N.$$

$\phi^{(i)}$  is computed using the integrals in (11). The  $(i+1)^{th}$  row of  $D_\psi$  is then computed as

$$(D_\psi)_{i+1,j} = \begin{cases} \sum_{r=1}^i \binom{i}{r} (D_\psi)_{r,j} (D_\phi)_{i+1-r,j} & i \leq k_j \\ 0 & \text{otherwise} \end{cases}, \quad j = 1, 2, \dots, N.$$

4. Compute the  $j^{th}$  Post-Widder approximation:

$$P_j = \frac{(-1)^{k_j-1}}{(k_j-1)!} \left(\frac{k_j}{x}\right)^{k_j} (D_\psi)_{k_j,j}.$$

5. If  $j = 1$ , set  $f_1 = P_1$ ,  $j = 2$ , and go to step 3. Otherwise, compute the  $j^{th}$  extrapolation as

$$f_j = \sum_{r=1}^j c_r P_r, \quad \text{where} \quad c_r = \prod_{\ell=1, \ell \neq r}^r \frac{k_r}{k_\ell - k_r}.$$

Set  $\alpha = 1 + 2(k_j/k_1 + 1)^{-1}$ . If  $|(1+\alpha)(f_j - f_{j-1})|/f_j < \epsilon$  or if  $j = N$ , return  $f_j$ . Otherwise, set  $j = j + 1$  and go to step 3.

The method for computing the CDF is similar, except that (12) is used instead for step 4.

**Algorithm for computing the CDF  $F$  at  $x > 0$**

Follow steps for PDF computation above except replace step 4. with

4.\* Compute the  $j^{th}$  Post-Widder approximation:

$$P_j = \sum_{r=0}^{k_j-1} \frac{(-1)^{k_j+r-1}}{(k_j-r-1)!} \left(\frac{k_j}{x}\right)^{k_j-r-1} (D_\psi)_{k_j-r,j}$$

We'll conclude this section with a collection of remarks regarding implementing this procedure.

### Remarks

1. Binomial coefficients are reused multiple times in step 3. We found it useful to compute and store rows of Pascal's triangle as needed.
2. In MATLAB, all computations above can be "vectorized" to maximize speed.
3. It is essential to compute the derivatives of  $\phi$  to as many *significant* digits of accuracy in step 3 as you want in your final result. In many examples, a closed form for  $\phi^{(n)}$  can be found. See Section 5 for examples where these integrals are computed without a closed form.

4. To avoid overflow, ratios of large numbers like those seen in steps 4. and 4.\* should be computed to incorporate reduction. For example, using identities such as

$$\frac{1}{(k_j - 1)!} \left( \frac{k_j}{x} \right)^{k_j} = \exp \left( k_j \log(k_j/x) - \sum_{j=1}^{k_j-1} \log(j) \right)$$

will alleviate overflow.

5. In double precision arithmetic, it is often impossible to take  $k_N$  much bigger than 100 as underflow, overflow and numerical instability become unavoidable. In most cases, adequate convergence is met much before such large  $k$  values are needed, however if one must go further, it will become necessary to work with a high-precision arithmetic package. *Mathematica*, for example, has powerful multiprecision capabilities. If one requires more speed, a well developed and documented library for multiple precision arithmetic for C++/Fortran is available at <http://crd.lbl.gov/~dhbailey/mpdist>.
6. When the true value of  $f$  is very close to 0 ( $f(x) \ll 10^{-8}$ ) the extrapolation procedure might “overshoot” 0 and return a negative value for the density. In this case, 0 is a good approximation as you can be sure that the PDF or CDF takes an extremely small value in this case. And similarly when the CDF takes a value greater than 1.

## 4 Testing the method

In this section we use *Mathematica* and consider two examples for which it is possible to compute the PDF and CDF exactly. The first example is the chi-squared distribution  $\chi^2$  with one degree of freedom. For this distribution,

$$\begin{aligned} f_{\chi^2}(x) &= \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2}, \quad F_{\chi^2}(x) = \operatorname{erf} \left( \sqrt{\frac{x}{2}} \right) \\ \psi_{\chi^2}(\lambda) &= \tilde{f}_{\chi^2}(\lambda) = (1 + 2\lambda)^{-1/2} \\ \phi_{\chi^2}(\lambda) &= -\log \psi_{\chi^2}(\lambda) = \frac{1}{2} \log(1 + 2\lambda) = \int_0^\infty (1 - e^{-\lambda u}) \left[ \frac{e^{-u/2}}{2u} \right] du. \\ \phi_{\chi^2}^{(n)}(\lambda) &= (-1)^{n+1} \int_0^\infty u^n e^{-\lambda n} \left[ \frac{e^{-u/2}}{2u} \right] du \\ &= \frac{(-1)^{n+1}}{2} \int_0^\infty u^{n-1} e^{-u(\lambda+1/2)} du = \frac{(-1)^{n+1}(n-1)!}{2} \left( \frac{1}{2} + \lambda \right)^{-n}, \quad n \geq 1. \end{aligned}$$

Results over a range of inputs is shown in Table 1. We show the  $x$ -values considered as well as the value of  $N$  required to obtain a relative precision of  $10^{-6}$  and  $10^{-15}$ . In Figure 2, we also plot the relative error as a function of  $N$ . Notice that the value of  $N$  required is substantially higher when the value of the PDF or CDF takes on very small values, in particular, when  $f_{\chi^2}(50) \approx 7.83 \times 10^{-13}$ . This is because  $f_k(x)$  over approximates  $f$  at these points.

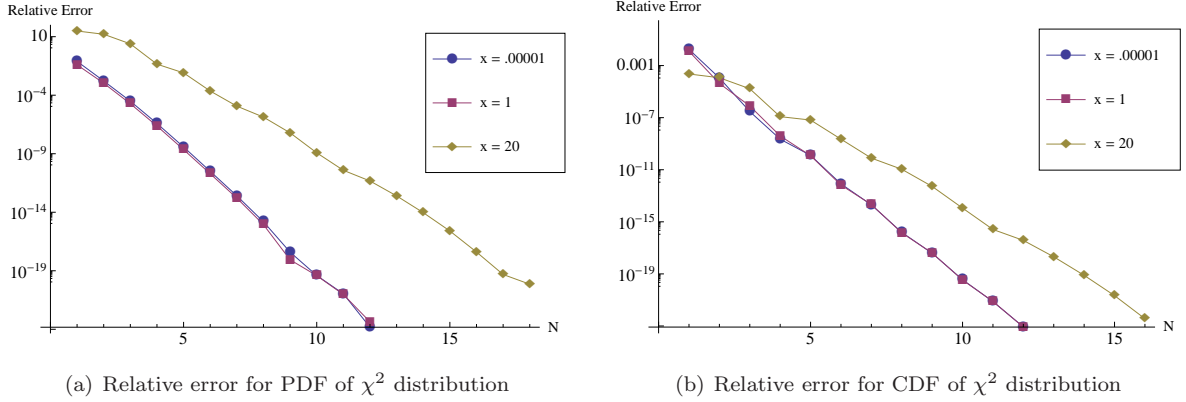


Figure 2: Plot of relative error of our method versus the truncation level  $N$  for the PDF and CDF of the  $\chi^2$  distribution at three values of  $x$ .

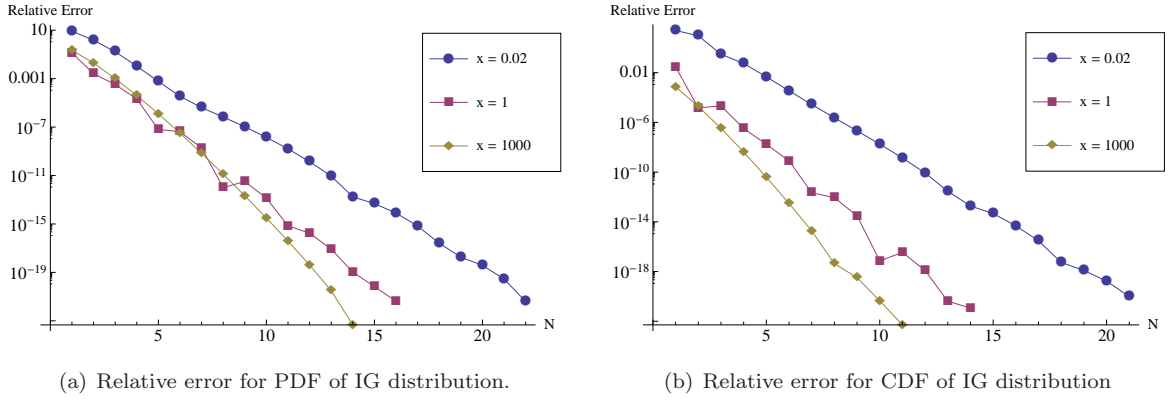


Figure 3: Plot of relative error of our method versus the truncation level  $N$  for the PDF and CDF of the IG distribution at three values of  $x$ .

The second example we considered is the inverse Gaussian (IG) distribution ([3], Example 1.3.21), for which

$$\begin{aligned}
 f_{IG}(x) &= \frac{1}{\sqrt{4\pi x^3}} e^{-1/(4x)}, \quad F_{IG}(x) = 1 - \operatorname{erf}\left(\frac{1}{\sqrt{2x}}\right) \\
 \psi_{IG}(\lambda) &= \tilde{f}_{IG}(\lambda) = e^{-\sqrt{\lambda}} \\
 \phi_{IG}(\lambda) &= -\log \psi_{IG}(\lambda) = \sqrt{\lambda} = \int_0^\infty (1 - e^{-\lambda u}) \left[ \frac{u^{-3/2}}{2\sqrt{\pi}} \right] du \\
 \phi_{IG}^{(n)}(\lambda) &= \frac{(-1)^{n+1}}{2\sqrt{\pi}} \int_0^\infty u^{n-3/2} e^{-\lambda u} du = \frac{(-1)^{n+1}}{2\sqrt{\pi}} \Gamma(n-1/2) \lambda^{1/2-n} = \frac{(-1)^{n+1} (2n-3)!}{2^{2(n-1)} (n-2)!} \lambda^{1/2-n}
 \end{aligned}$$

Table 1 also shows the values of  $x$  considered, the value of  $f_{IG}(x)$ , and the value of  $N$  needed to obtain relative errors of  $10^{-6}$  and  $10^{-15}$ . Figure 3 shows how the relative error behaves as a function of  $N$  for three values of  $x$ . Similarly to the  $\chi^2$  distribution,  $N$  is largest when the value of the PDF is very close to 0.

Chi-squared						
$x$	$10^{-5}$	$10^{-1}$	1	10	20	50
$f_{\chi^2}(x)$	$1.26 \times 10^2$	1.20	0.242	$8.50 \times 10^{-4}$	$4.05 \times 10^{-6}$	$7.83 \times 10^{-13}$
$N_6$	4	4	4	6	9	14
$N_{15}$	8	8	9	12	15	21
$F_{\chi^2}(x)$	$2.52 \times 10^{-3}$	0.248	0.683	0.998	$1 - 7.7 \times 10^{-6}$	$1 - 1.54 \times 10^{-13}$
$N_6$	3	3	3	4	4	1
$N_{15}$	8	8	8	11	11	12
Inverse Gaussian						
$x$	0.01	0.02	0.1	1	100	1000
$f_{IG}(x)$	$3.92 \times 10^{-9}$	$3.71 \times 10^{-4}$	0.732	0.220	$2.81 \times 10^{-4}$	$8.92 \times 10^{-6}$
$N_6$	12	8	6	5	6	6
$N_{15}$	21	17	14	11	10	11
$F_{IG}(x)$	$1.53 \times 10^{-12}$	$5.733 \times 10^{-7}$	0.025	0.480	0.944	0.982
$N_6$	12	9	6	4	4	3
$N_{15}$	21	17	13	10	8	8

Table 1: Using  $k_i = 10i$ , this table shows the value of  $N$  required to obtain 6 digits of accuracy ( $N_6$ ) and 15 digits of accuracy ( $N_{15}$ ) at the given values of  $x$  using polynomial interpolation. Notice that it becomes harder to approximate these functions to a small relative error when they become very small.

## 5 Applications to ID distributions which are not known in closed form

We will now consider examples of non-negative ID distributions for which the PDF and CDF are not known in closed form. We will compute them numerically using our method. The resulting plots of the PDFs and CDFs are obtained using MATLAB. Each plot was generated in about 1 second. In order to apply our method, we must first write the Laplace transform in LK form, and then find  $\phi(\lambda)$  and the derivatives  $\phi^{(n)}(\lambda)$  by computing the integrals in (11). In almost every case below, these integrals can be given in “closed form”, which is to say they can at least be written in terms of special functions for which there are efficient methods for computation. We will assume throughout that the drift  $a = 0$  in (2).

The following special functions will appear throughout this section, we provide their definitions here for convenience:

Gamma function	$\Gamma(a) = \int_0^\infty z^{a-1} e^{-z} dz, \quad a \neq 0, -1, -2, \dots$
Lower incomplete gamma function	$\gamma(a, b) = \int_0^b z^{a-1} e^{-z} dz, \quad b \geq 0, a \neq 0, -1, -2, \dots$
Upper incomplete gamma function	$\Gamma(a, b) = \int_b^\infty z^{a-1} e^{-z} dz, \quad b > 0, a \in \mathbb{R}$
Entire exponential integral	$\text{Ein}(a) = \int_0^a \frac{1 - e^{-z}}{z} dz, \quad a \in \mathbb{R}$
Dilogarithm	$L_2(a) = \int_1^a \frac{\log(z)}{z-1} dz, \quad a > 0$

See [15], chapters 25, 37, 43 and 45 for more discussion of these functions and methods for efficient compu-

tation. For an integer  $n \geq 1$ , we have  $\Gamma(n) = (n-1)!$  and

$$\gamma(n, b) = (n-1)! \left( 1 - e^{-b} \sum_{m=0}^{n-1} \frac{b^m}{m!} \right), \quad (21)$$

see [9], Equation 3.351.1.

## 5.1 Right-skewed stable distributions

Here we consider the class of examples with Laplace exponent given by

$$\phi(\lambda) = \int_0^1 \lambda^\beta p(d\beta) = \int_0^\infty (1 - e^{-\lambda u}) \left[ \int_0^1 \frac{u^{-\beta-1}}{\Gamma(-\beta)} p(d\beta) \right] du, \quad (22)$$

where  $p$  is a measure supported on  $(0, 1)$ . The Lévy measure  $\Pi$  has a density is given by

$$\Pi'(u) = \int_0^1 \frac{u^{-\beta-1}}{\Gamma(-\beta)} p(d\beta).$$

Mixtures of this form are considered in [13] and [12], in which these distributions are used in models of anomalous diffusion.

The Laplace exponent is expressed in (22) and the derivatives  $\phi^{(n)}$  for  $n \geq 1$  can be computed by using (11), a change in the order of integration, and the definition of the gamma function:

$$\phi^{(n)}(\lambda) = (-1)^{n+1} \int_0^\infty u^n e^{-\lambda u} \left( \int_0^1 \frac{u^{-\beta-1}}{\Gamma(-\beta)} p(d\beta) \right) du = (-1)^{n+1} \int_0^1 \frac{\lambda^{\beta-n} \Gamma(n-\beta)}{\Gamma(-\beta)} p(d\beta) \quad n \geq 1.$$

Since  $n$  takes only integer values,  $\Gamma(n-\beta) = \Gamma(-\beta) \prod_{m=0}^{n-1} (m-\beta)$ , and so the above can be simplified further as

$$\begin{aligned} \phi^{(n)}(\lambda) &= (-1)^{n+1} \int_0^1 \frac{\lambda^{\beta-n} \Gamma(n-\beta)}{\Gamma(-\beta)} p(d\beta) = (-1)^{n+1} \int_0^1 \lambda^{\beta-n} \prod_{m=0}^{n-1} (m-\beta) p(d\beta) \\ &= -\frac{1}{\lambda^n} \sum_{m=1}^n S_n^{(m)} \int_0^1 \beta^m \lambda^\beta p(d\beta) \\ &= -\frac{1}{\lambda^n} \sum_{m=1}^n c_m(\lambda) S_n^{(m)} \end{aligned} \quad (23)$$

where  $c_m(\lambda) = \int_0^1 \beta^m \lambda^\beta p(d\beta)$  and  $\{S_n^{(m)}\}$ ,  $n \geq 0$ ,  $0 \leq m \leq n$  are the Stirling numbers of the first kind ([15], page 162). These are such that  $\prod_{m=0}^{n-1} (\beta - m) = \sum_{m=1}^n S_n^{(m)} \beta^m$ , and can be computed with a triangular array similarly to Pascal's triangle using the recursion formula

$$\begin{aligned} S_0^{(0)} &= 1, \quad S_n^{(0)} = 0, \quad n \geq 1 \\ S_n^{(m)} &= S_{n-1}^{(m-1)} - (n-1) S_{n-1}^{(m)}, \quad n, m \geq 1. \end{aligned}$$

Let us now consider special cases of  $p$  (below,  $\delta$  denotes the dirac  $\delta$ -distribution).

**5.1.1 Right-skewed  $\alpha$ -stable distributions:**  $p(d\beta) = \delta(\beta - \alpha)d\beta$ ,  $0 < \alpha < 1$

Indeed, an important example of this distribution is the *right skewed  $\alpha$ -stable distributions*, for which  $p$  is a point mass at  $\beta = \alpha$ , with  $0 < \alpha < 1$  (using Proposition 1.2.11 in [20], this distribution corresponds to the stable distribution  $S(\alpha, 1, \cos(\pi\alpha/2)^{1/\alpha}, 0)$ ). These distributions lie in the family of scaling limits for sums of non-negative i.i.d. random variables with infinite mean. Form (23),  $\phi^{(n)}$  can be computed in various ways:

$$\phi^{(n)}(\lambda) = (-1)^{n+1} \frac{\lambda^{\alpha-n} \Gamma(n-\alpha)}{\Gamma(-\alpha)} = -\lambda^{\alpha-n} \prod_{m=0}^{n-1} (\alpha - m) = -\lambda^{\alpha-n} \sum_{m=1}^n \alpha^m S_n^{(m)}$$

The PDFs and CDFs of the right skewed  $\alpha$ -stable distribution are plotted using our method in Figure 4 for several values of  $\alpha$ . For example, we compute the PDF of a (1/2)-stable distribution at  $x = 1$  and obtain (in Mathematica)

$$f(1) = 0.219695644733861$$

This is exact to 15 decimal places and the computation took approximately half a second. An alternative method for this case is given in [14].

**5.1.2 Sums of right-skewed  $\alpha$ -stable distributions:**  $p(d\beta) = \sum_{j=1}^r d_j \delta(\beta - \alpha_j) d\beta$ ,  $d_j \geq 0$ ,  $\sum_{j=1}^r d_j = 1$

The previous case can easily be generalized to weighted sums of independent  $\alpha$ -stable random variables. From (23) it follows that in this case,

$$\phi^{(n)}(\lambda) = (-1)^{n+1} \sum_{j=1}^r d_j \frac{\lambda^{\alpha_j-n} \Gamma(n-\alpha_j)}{\Gamma(-\alpha_j)} = -\sum_{j=1}^r d_j \lambda^{\alpha_j-n} \prod_{m=0}^{n-1} (\alpha_j - m) = -\sum_{j=1}^r d_j \lambda^{\alpha_j-n} \left( \sum_{m=1}^n \alpha_j^m S_n^{(m)} \right).$$

Plot of PDFs of such distributions can be seen in Figure 5.

**5.1.3 A “uniform mixture” of  $\alpha$ -stable distributions:**  $p(d\beta) = d\beta$ ,  $\beta \in (0, 1)$

This is an example of a distribution with no finite moments. The Laplace exponent (22) is given by

$$\phi(\lambda) = \int_0^1 \lambda^\beta d\beta = \int_0^1 e^{\beta \log \lambda} d\beta = \frac{(\lambda - 1)}{\log \lambda} \quad \text{if } \lambda \neq 1,$$

and is 1 if  $\lambda = 1$ . Higher derivatives of  $\phi$  can be computed using (23). The coefficients  $c_m(\lambda)$ ,  $m = 1, 2, \dots, n$  can be computed in a few different ways. First, if  $\log \lambda \leq 0$ , we have

$$c_m(\lambda) = \int_0^1 \beta^m \lambda^\beta d\beta = \int_0^1 \beta^m e^{\beta \log \lambda} d\beta = \frac{\gamma(1+m, -\log \lambda)}{(-\log \lambda)^{m+1}} \quad (24)$$

where  $\gamma(a, b)$  is the lower incomplete gamma function (see above). The formula (24) can still be used for  $\log \lambda > 0$ , however this requires analytic continuation of  $\gamma$  which is not always easy to compute.

As another approach, we compute  $c_m(\lambda)$  by treating the cases  $|\log \lambda| < 1$  and  $\geq 1$  separately. If  $|\log \lambda| < 1$  notice that

$$c_m(\lambda) = \int_0^1 \beta^m e^{\log(\lambda)\beta} d\beta = \sum_{j=0}^{\infty} (\log \lambda)^j \int_0^1 \frac{\beta^{m+j}}{j!} d\beta = \sum_{j=0}^{\infty} \frac{(\log \lambda)^j}{(m+j+1)j!}. \quad (25)$$

If  $|\log \lambda| < 1$ , taking the first 18 terms in this series gives an absolute error of  $< 10^{-15}$ . For  $|\log \lambda| \geq 1$ , the  $c_m$ 's can be computed recursively by applying integration by parts:

$$\begin{aligned} c_0(\lambda) &= \int_0^1 e^{\log(\lambda)\beta} d\beta = \frac{\lambda - 1}{\log(\lambda)} \\ c_m(\lambda) &= \int_0^1 \beta^m e^{\log(\lambda)\beta} d\beta = \frac{\lambda - mc_{m-1}(\lambda)}{\log(\lambda)}, \quad m = 1, \dots, n. \end{aligned}$$

Since this procedure involves a division by  $\log \lambda$ , it should only be used when  $|\log \lambda| \geq 1$ .

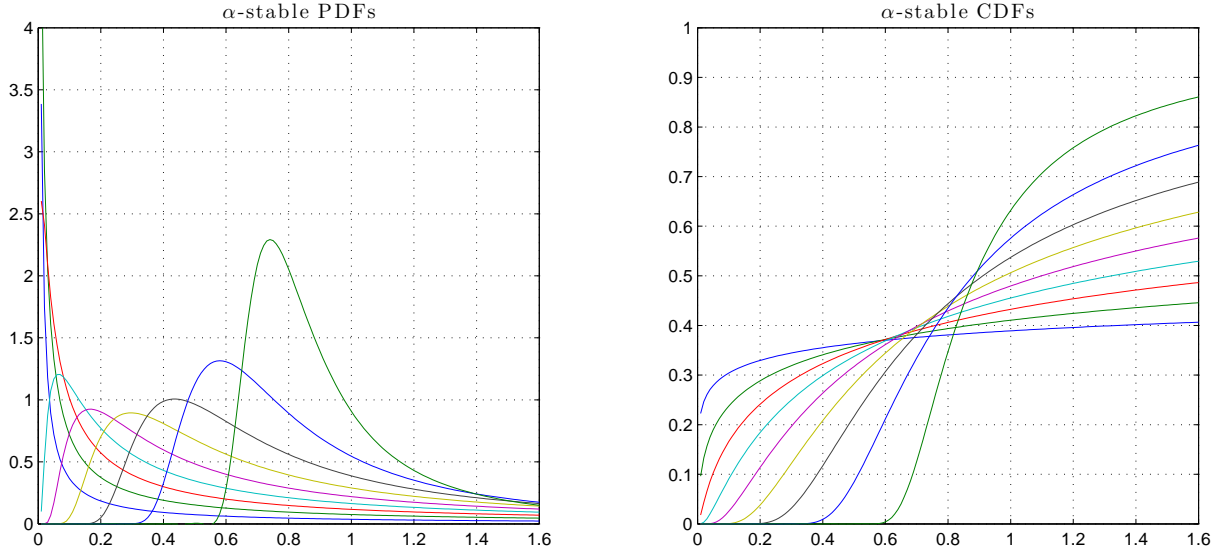


Figure 4:  $\alpha$ -stable PDFs (left) and CDFs (right) for  $\alpha = i/10$  for  $i = 1, 2, \dots, 9$ . The PDF for  $\alpha = 0.9$  corresponds to the right-most peak, and  $\alpha = .1$  corresponds to the left-most peak (which isn't visible). The CDF with the steepest slope around  $x = 0.8$  corresponds to  $\alpha = 0.9$ .

## 5.2 Integrals of non-random functions with respect to a Poisson random measure

In this section we consider the large class of non-negative ID distributions which can be expressed in terms of a Poisson stochastic integral of a non random kernel. For a recent review of these integrals, see [16].

Let  $(\Omega, \mathcal{F})$  be a probability space let  $\mathcal{B}$  denote the Borel sigma field on  $\mathbb{R}^n$ . Let  $N(\cdot)$  denote a independently scattered Poisson random measure with control measure  $\mu$ , that is,  $\mu$  is a measure on  $\mathcal{B}$  and  $N$  is a function  $\Omega \times \mathcal{B} \rightarrow \mathbb{Z}^+ \cup \{0\}$  such that

(i)  $N(A) \perp N(B)$  if  $A, B \in \mathcal{B}$  are disjoint.

(ii)  $N(\bigcup_{m=1}^n A_i) = \sum_{m=1}^n N(A_i)$  if  $A_i$  are disjoint.

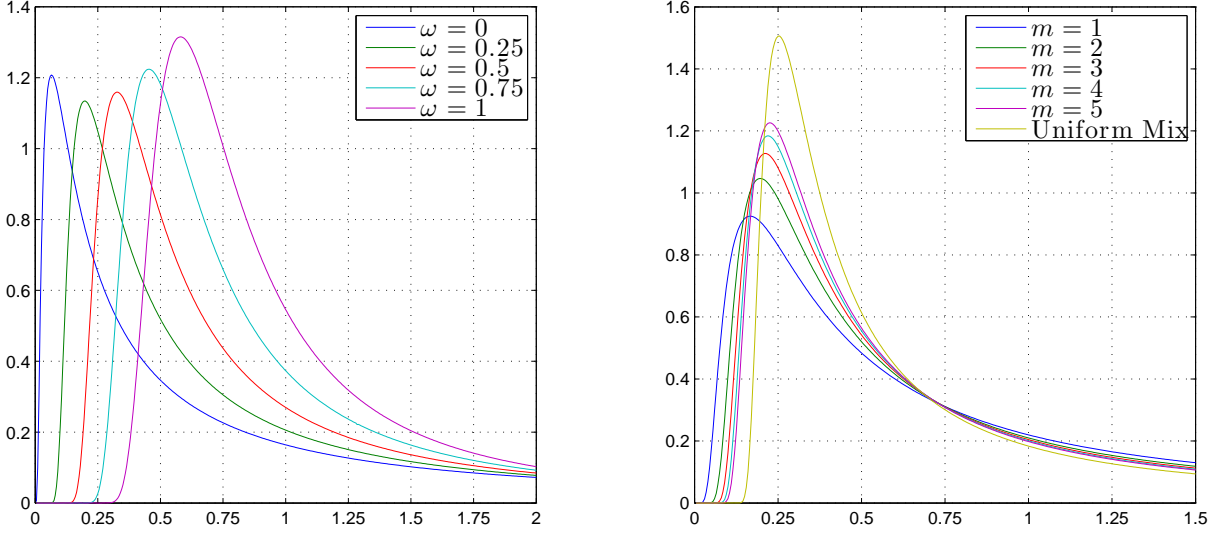


Figure 5: (Left figure) This is the plot of the PDF corresponding to the choice  $p_\omega(du) = (1 - \omega)\delta(x - 0.4) + \omega\delta(x - 0.8)$  for various choices of  $0 \leq \omega \leq 1$ . By increasing  $\omega$ , the PDF for  $\alpha = 0.4$  (on the left) is morphing into the PDF for  $\alpha = 0.8$  (on the right). (Right figure) These are PDFs corresponding to the sums  $p_m(du) = \frac{1}{m} \sum_{i=1}^m \delta(x - \frac{i}{m+1})$  for  $m = 1, 2, \dots, 5$  together with the uniform continuous mixture on  $(0, 1)$ , which has the highest peak.

(iii) For each  $A \in \mathcal{B}$ ,  $N(A)$  has a Poisson distribution with mean  $\mu(A)$ .

Given such a pair  $(N, \mu)$  one can define the following stochastic integral

$$I(g) = \int_{\mathbb{R}^n} g(\mathbf{x}) N(d\mathbf{x}) \quad (26)$$

for a function  $g$  on  $\mathbb{R}^n$ , for which  $\int_{\mathbb{R}^n} \min(1, g(\mathbf{x})) \mu(d\mathbf{x}) < \infty$  and, for our purposes, is non-negative. In this case, the random variable  $I(g)$  is also non-negative and has Laplace transform

$$\mathbb{E} e^{-\lambda I(g)} = \exp \left( - \int_{\mathbb{R}^n} (1 - e^{-\lambda g(\mathbf{s})}) \mu(d\mathbf{s}) \right), \quad \lambda > 0. \quad (27)$$

In order to compute the PDF and CDF of  $I(g)$  using our method, (27) must first be rewritten in LK form. In many cases, this can be done with some suitable change of variables  $\mathbf{u} = \Phi(\mathbf{s})$  satisfying  $u_1 = (\Phi(\mathbf{s}))_1 = g(\mathbf{s})$ . In this case, (27) becomes

$$\mathbb{E} e^{-\lambda I(g)} = \exp \left( - \int_0^\infty (1 - e^{-\lambda u_1}) \Pi_g(du_1) \right) \quad (28)$$

where for any Borel set  $A \subset \mathbb{R}^+$ , the Lévy measure  $\Pi_g(A)$  can be expressed formally as

$$\Pi_g(A) = \int_{A \cap \mathbb{R}^{n-1}} |J(u_1, u_2, \dots, u_n)| (\mu \circ \Phi^{-1})(du_1, du_2, \dots, du_n), \quad (29)$$



where  $|J|$  is the Jacobian  $\partial\Phi^{-1}(u_1, \dots, u_n)/\partial(u_1, \dots, u_n)$ .

To illustrate this, let us now focus on special cases, in dimensions  $n = 1$  and  $n = 2$ , where this change of variables can be made and our method applied.

### 5.2.1 One dimensional Poisson integral

Assume  $n = 1$  and that the integrand  $g$  is a monotone, non-negative function with inverse  $g^{-1}$ . In this case, (27) can be rewritten in LK form using the change of variables  $u = g(s)$ :

$$\mathbb{E}e^{-\lambda I(g)} = \exp\left(-\int_0^\infty (1 - e^{-\lambda u})\Pi_g(du)\right).$$

Since  $u = g^{-1}(g(u))$ , we get  $1 = (g' \circ g^{-1})(u) (g^{-1})'(u)$  and hence,

$$\Pi_g(du) = \frac{\mathbf{1}_{g((0,\infty))}(u)}{|(g' \circ g^{-1})(u)|}(\mu \circ g^{-1})(du), \quad (30)$$

where  $g((0, \infty))$  denotes the image of  $(0, \infty)$  under  $g$ . Suppose that we want to get the PDF and CDF of

$$I(g) = \int_0^\infty e^{-s/\eta} N(ds)$$

where  $g(s) = e^{-s/\eta}$ ,  $\eta > 0$  is a parameter and the control measure  $\mu$  is Lebesgue. In this case, a simple calculation shows that (30) becomes

$$\Pi_g(du) = \left(\frac{\eta}{u}\right) \mathbf{1}_{(0,1]}(u) du.$$

Thus, for this example,

$$\phi(\lambda) = \int_0^1 (1 - e^{-\lambda u}) \left(\frac{\eta}{u}\right) du = \eta \text{Ein}(\lambda),$$

where  $\text{Ein}(\lambda)$  is the entire exponential integral defined earlier. The derivatives  $\phi^{(n)}(\lambda)$ , for  $n \geq 1$  can be given in closed form

$$\phi^{(n)}(\lambda) = (-1)^{n+1} \eta \int_0^1 u^{n-1} e^{-\lambda u} du = \frac{(-1)^{n+1} \eta}{\lambda^n} \gamma(n, \lambda) = \frac{(-1)^{n+1} \eta}{\lambda^n} \left( (n-1)! - e^{-\lambda} \sum_{m=0}^{n-1} \frac{(n-1)!}{m!} \lambda^m \right),$$

where the last equality follows from (21). We've plotted the PDF and CDF of the random variable  $I(e^{-s/\eta})$  for various values of  $\eta$  in Figure (6) using our method. Note that since the range of integration here is  $(0, \infty)$ , the method described in [24] doesn't readily apply.

### 5.2.2 Integration with respect to non-negative Lévy process with a non-negative kernel

In this example, we generalize the previous case by looking at integration with respect to a non-negative Lévy process, or equivalently, a one-dimensional non-negative ID random measure  $L$  with control measure  $\mu$ .  $L$  is a random measure which satisfies the same conditions as the Poisson random measure  $N$ , except condition (iii) is replaced by

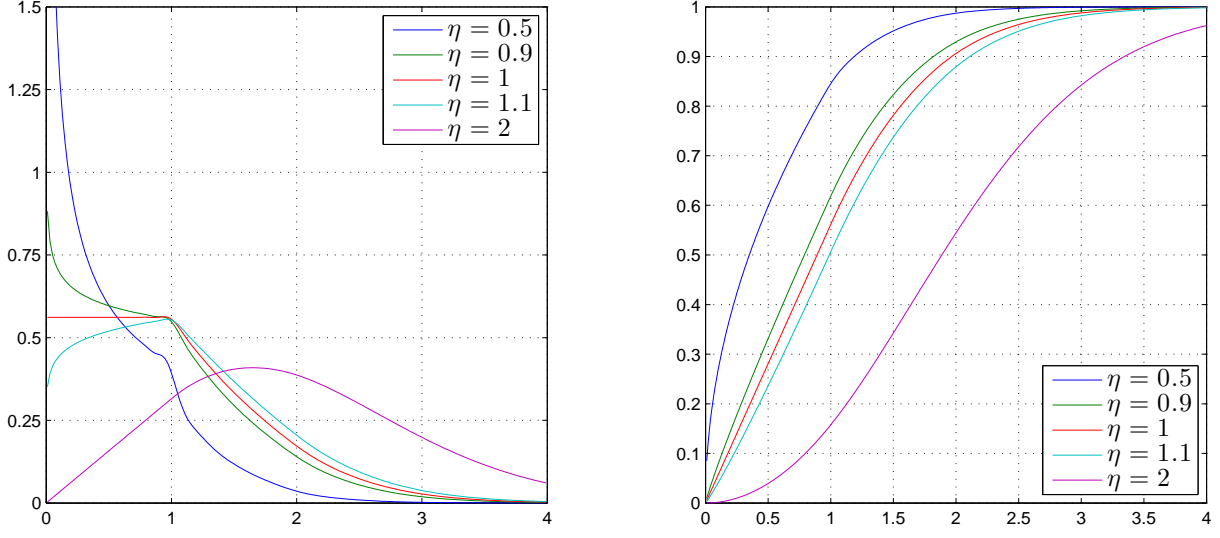


Figure 6: Plots of the PDF and CDF for the Poisson stochastic integral  $I(e^{-s/\eta})$  with various values of the parameter  $\eta$ , which plays the role of a shape parameter. The PDF for  $\eta = 0.5$  is the highest on the left.

(iii)' There exists a Lévy measure  $\Pi$  such that for any  $A \in \mathcal{B}$ , the distribution of  $L(A)$  has Laplace transform

$$\mathbb{E}e^{-\lambda L(A)} = \exp\left(-\mu(A) \int_0^\infty (1 - e^{-\lambda u})\Pi(du)\right).$$

Notice that the Poisson random measure  $N$  corresponds to the choice  $\Pi(du) = \delta(u-1)du$ . For a given function  $g$ , the stochastic integral  $I_L(g)$  can be defined in terms of a two-dimensional Poisson stochastic integral:

$$I_L(g) = \int_0^\infty g(s)L(ds) \equiv \int_0^\infty \int_0^\infty ug(z)N(du, dz), \quad (31)$$

where the control measure of  $N$  is given now by  $\Pi(du)\mu(dz)$ . Observe that the kernel  $g$  must now satisfy  $\int_0^\infty \int_0^\infty \min(1, ug(z))\Pi(du)\mu(dz) < \infty$ .

Assume now for simplicity that  $\Pi(du) = \Pi(u)du$  for some function  $\Pi$  and that  $g$  is a non-negative monotone function with inverse  $g^{-1}$ . In this case, we can obtain the LK form corresponding to  $I_L$ :

$$\begin{aligned} \mathbb{E}e^{-\lambda I_L(g)} &= \exp\left(-\int_0^\infty \int_0^\infty (1 - e^{-\lambda ug(z)})\Pi'(u)du\mu(dz)\right) \\ &= \exp\left(-\int_0^\infty (1 - e^{-\lambda v})\Pi'_g(v)dv\right) \end{aligned}$$

where we have made the change of variables  $v = ug(z)$ ,  $z' = z$  and the measure  $\Pi_g(dv)$  is given by

$$\Pi'_g(v) = \int_0^\infty \Pi'\left(\frac{v}{g(z')}\right) \frac{1}{g(z')} \mu(dz'). \quad (32)$$

To demonstrate our method in this case, consider

$$I_L(g) = \int_0^\infty e^{-z/\eta} L(dz) = \int_0^\infty \int_0^\infty u e^{-z/\eta} N(du, dz)$$

where  $g(z) = e^{-z/\eta}$  with  $\eta > 0$ ,  $\mu$  Lebesgue, and  $\Pi(du) = \kappa u^{-1} e^{-u/\theta} du$ , which is the Lévy measure corresponding to the Gamma distribution with shape  $\kappa > 0$  and scale  $\theta > 0$  ([3], Example 1.3.22). With the change of variables  $w = ue^{z/\eta}$ , (32) implies

$$\Pi'_g(u) = \int_0^\infty \Pi'(ue^{z/\eta}) e^{z/\eta} dz = \frac{\eta\kappa}{u} \int_u^\infty w^{-1} e^{-w/\theta} dw = \frac{\eta\kappa}{u} \Gamma\left(0, \frac{u}{\theta}\right)$$

We can now compute the corresponding Laplace exponent for this case:

$$\begin{aligned} \phi(\lambda) &= \int_0^\infty (1 - e^{-\lambda u}) \Pi'_g(u) du = \eta\kappa \int_0^\infty \frac{(1 - e^{-\lambda u})}{u} \Gamma\left(0, \frac{u}{\theta}\right) du \\ &= \eta\kappa \int_0^\infty \int_0^\lambda e^{-ut} \int_u^\infty (w^{-1} e^{-w/\theta}) dw dt du \\ &= \eta\kappa \int_0^\lambda \left[ \int_0^\infty \int_0^w (w^{-1} e^{-ut-w/\theta}) du dw \right] dt \\ &= \eta\kappa \int_0^\lambda \frac{1}{t} \left[ \int_0^\infty w^{-1} e^{-w/\theta} (1 - e^{-wt}) dw \right] dt \end{aligned} \quad (33)$$

$$\begin{aligned} &= \eta\kappa \int_0^\lambda \frac{1}{t} \int_0^\infty \left( e^{-w/\theta} \int_0^t e^{-ws} ds \right) dw dt \\ &= \eta\kappa \int_0^\lambda \frac{1}{t} \left[ \int_0^t \int_0^\infty e^{-w(s+1/\theta)} dw ds \right] dt \\ &= \eta\kappa \int_0^\lambda \frac{\log(1 + t\theta)}{t} dt = \eta\kappa L_2(1 + \lambda\theta) \end{aligned} \quad (34)$$

The derivatives of  $\phi$  can also be computed exactly in this case. Using (11) and (21),

$$\begin{aligned} \phi^{(n)}(\lambda) &= (-1)^{n+1} \eta\kappa \int_0^\infty u^{n-1} e^{-\lambda u} \Gamma\left(0, \frac{u}{\theta}\right) du \\ &= (-1)^{n+1} \eta\kappa \int_0^\infty u^{n-1} e^{-\lambda u} \int_u^\infty (w^{-1} e^{-w/\theta}) dw du \\ &= (-1)^{n+1} \eta\kappa \int_0^\infty w^{-1} e^{-w/\theta} \left[ \int_0^w u^{n-1} e^{-\lambda u} du \right] dw \\ &= (-1)^{n+1} \eta\kappa \int_0^\infty w^{-1} e^{-w/\theta} \left[ \frac{(n-1)!}{\lambda^n} - e^{-w\lambda} \sum_{m=0}^{n-1} \frac{(n-1)!}{m!} \frac{w^m}{\lambda^{n-m}} \right] dw \\ &= (-1)^{n+1} \eta\kappa \frac{(n-1)!}{\lambda^n} \left[ \int_0^\infty w^{-1} e^{-w/\theta} (1 - e^{-w\lambda}) dw - \sum_{m=1}^{n-1} \frac{\lambda^m}{m!} \int_0^\infty w^{m-1} e^{-w(\lambda+1/\theta)} dw \right] \\ &= (-1)^{n+1} \eta\kappa \frac{(n-1)!}{\lambda^n} \left[ \log(1 + \lambda\theta) - \sum_{m=1}^{n-1} \frac{(\lambda\theta)^m}{m(1 + \lambda\theta)^m} \right], \end{aligned}$$

where the first integral in the second to last line above is computed as in (33). Alternatively,  $\phi^{(n)}(\lambda)$  above can also be given simply in terms of the Gauss hypergeometric function  ${}_2F_1(\alpha, \beta, \gamma, x)$  ([9], Equation 6.455.1, page 657)

$$\phi^{(n)}(\lambda) = (-1)^{n+1} \eta \kappa \frac{\theta^n (n-1)!}{n} {}_2F_1(n, n, n+1, \lambda \theta).$$

In Figure 7 we've plotted the PDF and CDF of (31) for  $\theta = 1$  and various values of the product  $\rho = \eta \kappa$  using our method.

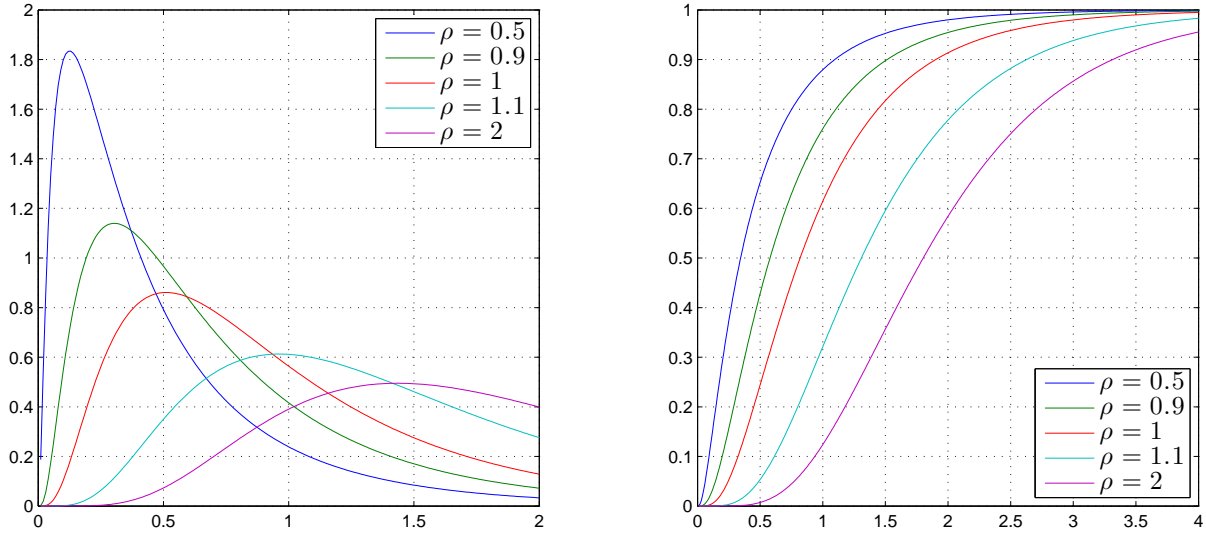


Figure 7: Plots of the PDF and CDF for the Poisson stochastic integral  $I_L(e^{-s/\eta})$  with various values of the parameter  $\eta$ .

## 6 Guide to software

In this section, we will explain how to use the software written to implement the method discussed in this paper. Versions of this code exist in MATLAB and Mathematica, and are freely available by request from the authors. Each version will include a file containing examples to assist in using the code.

To begin using the code, download the file `NNINFDIV.zip` and extract the directory `NNINFDIV`. This directory contains both the MATLAB and Mathematica programs in separate folders. We will now focus on these separately.

### 6.1 MATLAB version

To use the MATLAB version, launch MATLAB and add the directory `NNINFDIV/MATLAB` to MATLAB's default path by typing

```
>> path(path, 'mypath/NNINFDIV/MATLAB')
```

where 'mypath' is the path which leads to the directory NNINFDIV. You are now ready to use the code provided in this package.

The main function is called `nninfddiv`. This function takes in 5 arguments in the following order:

<code>X</code>	Scalar or vector of input values. Must be positive.
<code>DIST</code>	A cell array which specifies the distribution and parameters (see below)
<code>FUNC</code>	Type of function: 'pdf' or 'cdf'
<code>METHOD</code>	Extrapolation method: 'polynomial' or 'rational'
<code>TOL</code>	Target relative error tolerance

The last three inputs `FUNC`, `METHOD` and `TOL` are optional, and take default values 'pdf', 'polynomial' and  $10^{-6}$ . The input `DIST` is a cell array which contains the name of the desired distribution followed by the parameters. Possibilities for `DIST` are

<code>{'chi-squared',df,1}</code>	Chi-squared distribution with <code>df</code> degrees of freedom.
<code>{'chi-squared',df,[c1,...,cn]}</code>	Weighted sum of chi-squared distributions
<code>{'alpha stable',a,c}</code>	Alpha-stable distribution with $\alpha = a$ and scaling $c$ .
<code>{'alpha stable',[a1,...,an],[c1,...,cn]}</code>	Sum of $n$ weighted alpha-stable distributions.
<code>{'uniform mix'}</code>	Uniform mix from Section 5.1.3.
<code>{'ou poisson',eta}</code>	The integral $I(e^{-s/\eta})$ from Section 5.2.1
<code>{'ou gamma',eta,kappa}</code>	The integral $I_L(e^{-s/\eta})$ from Section 5.2.2.

The scaling constants seen in the alpha-stable and chi-squared examples above compute the PDF/CDF of the scaled random variables  $cX$  for  $c > 0$  in the single alpha-stable and single chi-squared case. Likewise, in the weighted case, they return the PDF/CDF of  $c_1X_1 + c_2X_2 + \dots c_nX_n$  with  $c_i > 0$ , and  $X_i$  is chi-squared with `df` degrees of freedom or is alpha-stable with  $\alpha = ai$ .

Since the input for `nninfddiv` is long, it is often useful to define a *function handle* in order to call the function more easily. For example, consider the  $\alpha$ -stable distribution with  $\alpha = 2/3$ . We define the PDF of this distribution in the variable `f` by typing

```
>> f = @(x) nninfddiv(x,{'alpha stable',2/3,1},'pdf','polynomial',1e-6);
```

The function `f` now computes the PDF of the alpha-stable distribution with  $\alpha = 2/3$  to within a relative error of  $10^{-6}$  using the polynomial interpolation method. For example, you may now type

<code>&gt;&gt; f(1)</code>	Computes the PDF at $x = 1$
<code>&gt;&gt; f([1 2 3])</code>	Computes the PDF at $x = 1, 2$ and $3$ .
<code>&gt;&gt; plot([.05:.05:2],f([.05:.05:2]))</code>	Plots the PDF on the interval $(0,2]$

**Remark:** Obtaining relative errors less than  $10^{-6}$  is sometimes difficult. If your error tolerance cannot be reached, the program will return the best estimate possible in double precision. If high precision is preferred over speed, the Mathematica version should be used.

## 6.2 Mathematica version

To use the Mathematica version, launch `Mathematica` open the file `NNINFDIV.nb` located in the directory `NNINFDIV/Mathematica`. Once this file is open, select all its contents by pressing `alt-a` on a PC or `cmd-a` on

a Mac. Then compile the code by pressing **shift-return**. You are now ready to use the code in a separate notebook.

The main program is called `NNInfDiv` (capitalization matters). This program is called with 4 arguments:

<code>X</code>	Input value. Must be a positive scalar.
<code>DIST</code>	A list which specifies the distribution and parameters (see below)
<code>FUNC</code>	Type of function: "PDF" or "CDF"
<code>TOL</code>	Relative error tolerance

The last two inputs `FUNC` and `TOL` are optional, taking default values "PDF" and  $10^{-15}$  respectively. Possibilities for `DIST` include

<code>{"Chi-Squared",{c1,..cn}}</code>	Sum of weighted chi-squared with weights $c_1, \dots, c_n$ .
<code>{"Alpha Stable",a,c}</code>	Alpha-stable distribution with $\alpha = a$ and scaling $c$ .
<code>{"Uniform Mix"}</code>	Uniform mix from Section 5.1.3.
<code>{"OU Poisson",eta}</code>	The integral $I(e^{-s/\eta})$ from Section 5.2.1
<code>{"OU Gamma",eta,kappa}</code>	The integral $I_L(e^{-s/\eta})$ from Section 5.2.2.

The scaling constants seen in the chi-squared and alpha-stable cases above refer to the random variables  $cX$  with  $c > 0$  in the alpha-stable case and  $c_1X_1 + \dots c_nX_n$  in the chi-squared case, with  $X_i$  i.i.d chi-squared.

To simplify the call to this function, one can make a user defined function. For example, to make a function `F` which computes the CDF of an  $\alpha$ -stable distribution with  $\alpha = 2/3$ , one can type

```
F[x_] := NNInfDiv[x,{ "Alpha Stable" , 2/3 , 1 } , "CDF" ]
```

The function `F` now computes the CDF of  $\alpha$ -stable distribution with  $\alpha = 2/3$  to a relative precision of  $10^{-15}$ . For example, one can now enter

<code>F[1]</code>	Computes the CDF at $x = 1$
<code>Table[F[x] , {x,{1,2,3}}]</code>	Computes the CDF at $x = 1, 2$ and $3$
<code>ListPlot[Table[{x,F[x]},{x,0,2,.05}],Joined -&gt; True]</code>	Plots the CDF on the interval $[0,2]$

**Remark:** Using `NNInfDiv` with Mathematica's `Plot` function is very slow, which is why we used the `ListPlot` function above. For faster plotting and function evaluation, the MATLAB version of the code should be used.

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